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Michael Frey† and Emil Simiu‡
†Department of Mathematics, Bucknell University, Lewisburg, PA 17837
‡Structures Division, Building and Fire Research Laboratory, National Institute of Standards and Technology, Gaithersburg, MD 20899

Noise-Induced Transitions to Chaos

INTRODUCTION

Multistable systems can exhibit irregular (i.e., neither periodic nor quasiperiodic) motion with jumps. Such motion is referred to as basin-hopping or stochastic chaos when induced by noise,[1] and deterministic chaos in the absence of noise. Deterministic and stochastic chaos have hitherto been viewed as distinct and have been analyzed from different, indeed contrasting, points of view.

For a wide class of systems, stochastic and deterministic chaos can be not only indistinguishable phenomenologically but also closely related mathematically. We show this for one-degree-of-freedom multistable systems whose unperturbed counterparts have homoclinic and heteroclinic orbits. (Extensions of the theory to higher-degree-of-freedom and spatially extended systems are underway.) When perturbed by weak damping and deterministic periodic forcing, the dynamics of these systems are periodic or quasiperiodic over certain regions of the system parameter space. Over other regions of parameter space the dynamics may be sensitively dependent upon initial conditions; i.e., exhibit a topological equivalence to the Smale horseshoe map. We show that a transition from periodic or quasiperiodic motion to chaotic motion with sensitive dependence upon initial conditions is possible through the introduction of noise.

We develop computable expressions providing: (1) necessary conditions for the occurrence of stochastic chaos with jumps, and (2) measures of chaotic transport characterizing the "intensity" of the chaos. These expressions depend on the distribution and, in particular, the mean-square spectrum of the noise. We obtain these expressions using: (1) the Melnikov transform and its attendant notion of phase space flux, and (2) tail-limited noise including uniformly bounded path approximations of Gaussian noise and shot noise.

The remainder of the chapter is divided into five sections. The first section presents results for systems perturbed by weak additive noise. These results are used in the following section to treat Duffing oscillators with weak near-Gaussian noise. In particular, we describe results for the Duffing oscillator with attracting homoclinic orbits which admit comparison with results based on the Fokker-Planck equation. In the third section, we present results for multiplicatively perturbed systems. These results are used in the fourth section with a recently introduced model of shot noise to treat the Duffing oscillator with shot noise-like dissipation. The last section contains summary comments.

SYSTEMS WITH ADDITIVE EXCITATION

We consider the integrable, two-dimensional, one-degree-of-freedom dynamical system with energy potential V governed by the equation of motion

$$\ddot{x} = -V'(x), \qquad x \in \mathcal{R}. \tag{1}$$

The system governed by Eq. (1) is assumed to have two hyperbolic fixed points connected by a heteroclinic orbit $\vec{x}_s = (x_s(t), \dot{x}_s(t))$. If the two hyperbolic fixed points coincide, then $\vec{x}_s$ is homoclinic. A perturbative component is introduced into the system governed by Eq. (1), giving

$$\ddot{x} = -V'(x) + \varepsilon w(x, \dot{x}, t). \tag{2}$$

The perturbative function $w : \mathcal{R}^2 \times \mathcal{R} \to \mathcal{R}$ is assumed to satisfy the Meyer-Sell uniform continuity conditions[15] and only the near-integrable case, $0 < \varepsilon \ll 1$, is considered. In this section, we restrict our attention to the case of additive excitation and linear damping. For this case,

$$w(x, \dot{x}, t) = \gamma g(t) + \rho G(t) - \kappa \dot{x} \tag{3}$$

and system (2) takes the form

$$\ddot{x} = -V'(x) + \varepsilon[\gamma g(t) + \rho G(t) - \kappa \dot{x}]. \tag{4}$$

Here g and G represent deterministic and stochastic forcing functions, respectively. g is assumed to be bounded, $|g(t)| \leq 1$, and uniformly continuous (UC). The parameters ρ, γ and κ are nonnegative and fix the relative amounts of damping and external forcing in the model.

Consider the random forcing

$$G(t) = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \frac{\sigma}{S(\nu_n)} \cos(\nu_n t + \varphi_n), \tag{5}$$

where $\{\nu_n, \phi_n; n = 1, 2, \ldots, N\}$ are independent random variables defined on a probability space $(\Omega, \mathcal{B}, P)$, $\{\nu_n; n = 1, 2, \ldots, N\}$ are nonnegative with common distribution Ψ_o, $\{\phi_n; n = 1, 2, \ldots, N\}$ are identically uniformly distributed over the interval $[0, 2\pi]$ and N is a fixed parameter of the model. S and σ in Eq. (5) are defined below. The process G is a randomly weighted modification of the Shinozuka noise model.[18,19]

Let $\mathcal{F}$ denote the linear filter with impulse response $h(t) = \dot{x}_s(-t)$ where $\dot{x}_s(t)$ is the velocity component of the orbit $\vec{x}_s$ of the system governed by Eq. (1). $\mathcal{F}$ is called the system orbit filter and its output is $\mathcal{F}[u] = u * h$ where $u = u(t)$ is the filter input and $u * h$ is the convolution of u and h. S in Eq. (5) is then defined to be modulus $S(\nu) = |H(\nu)|$ of the orbit filter transfer function

$$H(\nu) = \int_{-\infty}^{\infty} h(t) e^{-j\nu t} dt \tag{6}$$

and σ in Eq. (5) is

$$\sigma^2 = \int_0^{\infty} S^2(\nu) \Psi(d\nu).$$

Let the distribution Ψ_o of the angular frequencies ν_n in Eq. (5) have the form

$$\Psi_o(A) = \frac{1}{\sigma^2} \int_A S^2(\nu) \Psi(d\nu), \tag{7}$$

where A is any Borel subset of $\mathcal{R}$. S is assumed to be bounded away from zero on the support of Ψ, $S(\nu) > S_m > 0$ a.e. Ψ. Under this condition S is said to be Ψ-admissible. If S is Ψ-admissible, then it is also bounded away from zero on the support of Ψ_o and $1/S(\nu_n) < 1/S_m$ a.s. Ψ_o. We have the following results for G and its filtered counterpart $\mathcal{F}[G]$.

Fact G1. The processes G and $\mathcal{F}[G]$ are each zero-mean and stationary.

Fact G2. If S is Ψ-admissible then G is uniformly bounded with $|G(t,\omega)| \leq \sqrt{2N/S_m}$ for all $t \in \mathcal{R}$ and $\omega \in \Omega$.

Fact G3. The marginal distribution of $\mathcal{F}[G]$ is that of the sum

$$\sigma\sqrt{\frac{2}{N}}\sum_{n=1}^{N}\cos U_n$$

where $\{U_n; n = 1,\ldots,N\}$ are independent random variables uniformly distributed on the interval $[0, 2\pi]$.

Fact G4. The processes G and $\mathcal{F}[G]$ are each asymptotically Gaussian in the limit as $N \to \infty$. In particular, the random variables $G(t)$ and $\mathcal{F}[G](t)$ are, for each t, asymptotically Gaussian.

Fact G5. The spectrum of G is $2\pi\Psi$ and G has unit variance.

Fact G6. The spectrum of $\mathcal{F}[G]$ is $2\pi\Psi_o$ and its variance is σ^2.

Fact G7. Let the spectrum Ψ of G be continuous. Then $\mathcal{F}[G]$ is ergodic.

Proof of the first six of these results can be found in Frey and Simiu.[7] *Fact G7* is related to the fact that Gaussian processes with continuous spectra are ergodic.[10,14] It follows from *Fact G5* that a modified Shinozuka noise process G as in Eq. (5) can be constructed for any given spectrum.

Five realizations of G with bandlimited spectrum are shown for comparison in Figure 1 together with five realizations of Gaussian noise with the same spectrum. $S(\nu) = \mathrm{sech}\nu$ is used in this example.

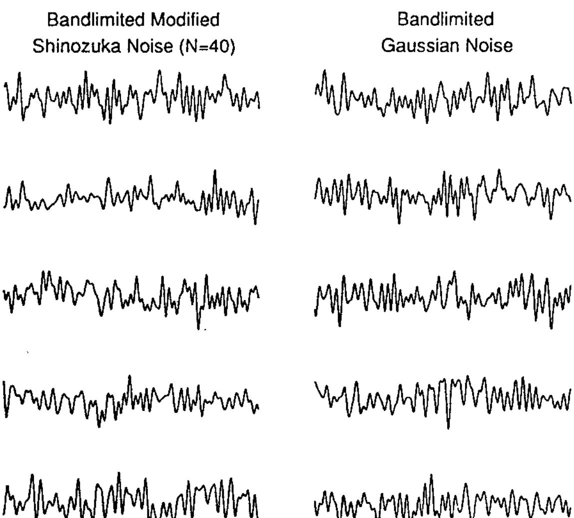

FIGURE 1 Realizations of modified Shinozuka and Gaussian noise processes with identical bandlimited spectra and $S(\nu) = \mathrm{sech}\nu$

Let us now consider the effect of the perturbation $\varepsilon w(x, \dot{x}, t)$ on the global geometry of Eq. (1). For sufficiently small perturbations, the hyperbolic fixed points of Eq. (1) are perturbed to a nearby invariant manifold and the stable and unstable manifolds associated with the homoclinic or heteroclinic orbit of Eq. (1) separate.[2] The distance between the separated manifolds is expressible as an asymptotic expansion $\varepsilon M + O(\varepsilon^2)$ where M is a computable quantity called the Melnikov function. The separated manifolds may intersect transversely and, if such intersections occur, they are infinite in number and define lobes marking the transport of phase space[24]. The amount of phase space transported, the phase space flux, is a measure of the chaoticity of the dynamics.[3] The lobes defined by the intersecting manifolds generally have twisted, convoluted shapes whose areas are difficult to determine, making analytical calculation of the flux difficult, if not impossible. For the case of small perturbations, however, the phase space flux can be expressed in terms of the Melnikov function. The average phase space flux has the asymptotic expansion $\varepsilon\Phi + O(\varepsilon^2)$ where Φ, here called the flux factor, is a time average of the Melnikov function:

$$\Phi = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} M^{+}(\theta_1 - t, \theta_2 - t)dt, \tag{8}$$

where M^+ is the maximum of 0 and M.[3,24]

To apply Melnikov theory to a deterministic excitation g, g must be bounded and UC. In the case of random perturbations G, the theory requires that G be uniformly bounded and uniformly continuous across both time and ensemble. The noise model G in Eq. (5) is uniformly bounded as noted in *Fact G2*. However, G does not necessarily have the needed degree of continuity.

We define a stochastic process X to be ensemble uniformly continuous (EUC) if, given any $\delta_1 > 0$, there exists $\delta_2 > 0$ such that if $t_1, t_2 \in \mathcal{R}$ and $|t_1 - t_2| < \delta_2$, then $|X_{t_1}(\omega) - X_{t_2}(\omega)| < \delta_1$ for all $\omega \in \Omega$. A stochastic process can have UC paths and fail to be EUC. G is EUC if it is bandlimited.

Conditions on the perturbation function w in Eq. (3) sufficient for the Melnikov function to exist are that g be UC and that G be EUC. The Melnikov function for the system governed by Eq. (4) is then given by the Melnikov transform $\mathcal{M}[g, G]$ of g and G:

$$\begin{aligned} M(t_1, t_2) =& \mathcal{M}[g, G] \\ =& -\kappa \int_{-\infty}^{\infty} \dot{x}_s^2(t)dt + \gamma \int_{-\infty}^{\infty} \dot{x}_s(t)g(t + t_1)dt \\ &+ \rho \int_{-\infty}^{\infty} \dot{x}_s(t)G(t + t_2)dt. \end{aligned} \tag{9}$$

Since $h(t) = \dot{x}_s(-t)$, denoting the integral of $\dot{x}_s^2$ by I, we obtain

$$M(t_1, t_2) = -I\kappa + \gamma\mathcal{F}[g](t_1) + \rho\mathcal{F}[G](t_2). \tag{10}$$

The expectation and variance of $M(t_1, t_2)$ are, respectively,

$$E[M(t_1, t_2)] = -I\kappa + \gamma\mathcal{F}[g](t),$$

$$Var[M(t_1, t_2)] = \rho^2\sigma^2 = \rho^2 \int_0^\infty S^2(\nu)\Psi(d\nu).$$

$M(t_1, t_2)$ is, like G, a Gaussian process in the limit as $N \to \infty$ indicating that the presence of even vanishingly small noise causes the Melnikov function to have simple zeros. The state of the system is thus driven from one basin of attraction to that of the competing attractor. Such motion is interpretable as chaotic motion on a single strange attractor.[24]

Of course, the infinitely long tails of the marginal distributions of Gaussian noise are physically unrealistic. Expressions for random forcing with tail-limited marginal distributions can be obtained through nonlinear transformations of Eq. (5). Such tail-limited excitation processes (which may represent, e.g., wave forces whose magnitude is limited by physical constraints) are of interest in engineering applications. From such noise models, the tail-limited marginal distributions of the Melnikov function can be determined, allowing criteria to be developed to guarantee that the Melnikov function has no simple zeros, i.e, that jumps (snap-through dynamics) associated with chaos do not occur.

We now proceed to develop formulae for the average flux factor Φ. Substituting Eq. (10) into Eq. (8) we obtain

$$\Phi = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [\gamma\mathcal{F}[g](\theta_1 - s) + \rho\mathcal{F}[G](\theta_2 - s) - I\kappa]^+ ds. \tag{11}$$

Existence of the limit in Eq. (11) depends on the nature of the excitations g and G and their corresponding convolutions $\mathcal{F}[g] = g * h$ and $\mathcal{F}[G] = G * h$.

To ensure the existence of the limit in Eq. (11), we assume that g is asymptotic mean stationary (AMS): a stochastic process $X(t)$ is defined to be AMS if[9] the limits

$$\mu_X(A) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[1_A(X(t))]dt \tag{12}$$

exists for each real Borel set $A \in \mathcal{R}$. Here 1_A is the indicator function, $1_A(x) = 1$ for $x \in A$ and $1_A(x) = 0$ otherwise. This definition applies, in particular, to deterministic functions $X(t)$. If the limits in Eq. (12) exist, then μ_X is a probability measure.[13] μ_X is called the stationary mean (SM) distribution of the process X.

The deterministic forcing function g is assumed to be AMS so, due to the linearity of $\mathcal{F}$, $\mathcal{F}[g]$ is AMS and we denote the SM distribution of $\mathcal{F}[g]$ by $\mu_{\mathcal{F}[g]}$. Assume the spectrum of G is continuous. Then, according to *Fact G7*, $\mathcal{F}[G]$ is ergodic. Ergodicity implies asymptotic mean stationarity,[9] so $\mathcal{F}[G]$ is AMS also with SM distribution $\mu_{\mathcal{F}[G]}$. All AMS deterministic functions are ergodic so $\mathcal{F}[g]$, like $\mathcal{F}[G]$, is ergodic. Inasmuch as $\mathcal{F}[g]$ is deterministic, $\mathcal{F}[g]$ and $\mathcal{F}[G]$ are jointly ergodic with SM distribution $\mu_{\mathcal{F}[g]} \times \mu_{\mathcal{F}[G]}$.[7] Then the limit Eq. (11) exists and can be expressed in terms of the SM distributions $\mu_{\mathcal{F}[g]}$ and $\mu_{\mathcal{F}[G]}$.

THEOREM 1. (See Frey and Simiu.[7]) Suppose g is AMS and $\mathcal{F}[G]$ is ergodic. Then the limit in Eq. (11) exists, the flux factor Φ is nonrandom and

$$\Phi = E[(\gamma A + \rho B - I\kappa)^+]$$

where A is a random variable with distribution equal to the SM distribution $\mu_{\mathcal{F}[g]}$ of the function $\mathcal{F}[g]$, B is a random variable with distribution equal to the SM distribution $\mu_{\mathcal{F}[G]}$ of the process $\mathcal{F}[G]$ and A and B are independent.

Theorem 1 applies broadly to uniformly bounded and EUC noise processes G with ergodic filtered counterpart $\mathcal{F}[G]$. The modified Shinozuka process (5) belongs to this class provided it is Ψ-admissible with continuous, bandlimited spectrum. Moreover, G in (5) is stationary and $\mathcal{F}[G]$ is asymptotically Gaussian. Hence $\mu_{\mathcal{F}[G]}$ is the marginal distribution of $\mathcal{F}[G]$ and, for large N, B is approximately Gaussian with zero mean and variance σ^2.

THEOREM 2. (See Frey and Simiu.[7]) Suppose g is AMS and G is a Ψ-admissible modified Shinozuka process with continuous bandlimited spectrum. Then the flux factor Φ is approximately

$$\Phi \doteq E[(\gamma A + \rho\sigma Z - I\kappa)^+] \tag{13}$$

where Z is a standard Gaussian random variable. The error in this approximation decreases as N is made larger.

DUFFING OSCILLATOR WITH ADDITIVE NEAR-GAUSSIAN NOISE

The potential energy of the Duffing oscillator is $V(x) = x^4/4 - x^2/2$ with corresponding equation of motion, $\ddot{x} = x - x^3$. The unperturbed Duffing oscillator has a hyperbolic fixed point at the origin $(x, \dot{x}) = (0,0)$ in phase space connected to itself by symmetric homoclinic orbits. These orbits are given by

$$\begin{pmatrix} x_s(t) \\ \dot{x}_s(t) \end{pmatrix} = \pm \begin{pmatrix} \sqrt{2}\operatorname{sech} t \\ -\sqrt{2}\operatorname{sech} t \tanh t \end{pmatrix}.$$

The impulse response h of the righthand (+) orbit is $h(t) = \dot{x}_s(-t) = \sqrt{2}\operatorname{sech} t \tanh t$. Thus $I = 4/3$ in Eq. (10).

OSCILLATOR WITH LINEAR DAMPING

We consider first the forced Duffing oscillator with additive noise and linear damping:

$$\ddot{x} = x - x^3 + \varepsilon[\gamma g(t) + \rho G(t) - \kappa \dot{x}]. \tag{14}$$

Here, $\gamma \geq 0$, $\kappa \geq 0$ and $\rho \geq 0$ are constants, g is deterministic and bounded $|g(t)| \leq 1$, and G is the modified Shinozuka noise process reviewed in the previous section.

The flux factor Φ for this system is given exactly in Theorem 1 and approximately in Theorem 2. The approximation in Theorem 2 was obtained by representing the marginal distribution $\mu_{\mathcal{F}[G]}$ of $\mathcal{F}[G]$ by a Gaussian distribution and is appropriate for large N. However, because the Gaussian distribution has infinite tails, Theorem 2 indicates that the flux factor is nonzero for all levels $\rho > 0$ of noise.

Consider the case $\gamma = 0$. According to Theorem 1, $\Phi = E[(\rho\sigma B_N - 4\kappa/3)^+]$. Using *Fact G3*, we take

$$B_N = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \cos U_n$$

and define

$$\Phi' = \frac{3\Phi}{4\kappa}, \quad \rho' = \frac{3}{\sqrt{2}} \frac{\rho\sigma}{\kappa}, \quad B' = \frac{B_N + \sqrt{2N}}{2\sqrt{2N}}.$$

Then

$$\Phi' = E[(\rho'\sqrt{N}(B' - 1/2) - 1)^+]. \tag{15}$$

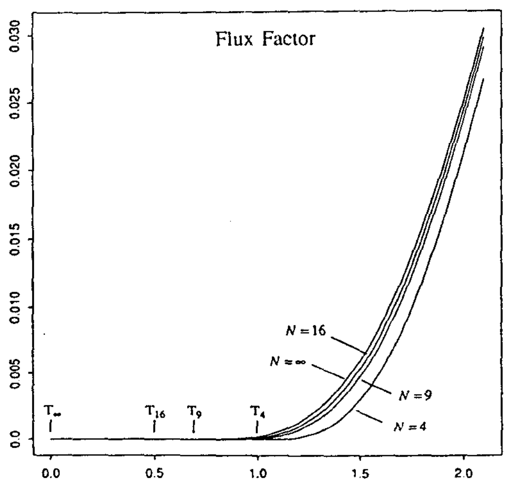

FIGURE 2 The flux factor of Φ' as a function of the noise strength ρ' for various values of N. T_N is the threshold for positive flux.

The support of B' is the interval $(0, 1)$ and is approximately beta-distributed[16] with density

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1}(1-t)^{\beta-1}, \quad 0 < t < 1$$

where the parameters $\alpha > 0$ and $\beta > 0$ of the distribution are chosen so that the mean and the variance of the beta distribution are the same as those of B'. $\Phi' = \Phi'(\rho', N)$ is plotted in Figure 2 as a function of ρ' for various values of N using Eq. (16). For comparison, the limiting Gaussian noise case $N \to \infty$ is also plotted using Eq. (13).

OSCILLATOR WITH ATTRACTING HOMOCLINIC ORBIT

If the damping term in Eq. (14) has the form $\epsilon(\kappa - \beta x^2)x$, under suitable conditions on the coefficients κ and β the homoclinic orbits of the unperturbed flow are attracting; i.e., among all orbits passing through a neighborhood of the homoclinic orbit, almost all can be found to be in the vicinity of the homoclinic orbit at times in the far future or past.[5] The dynamics of this oscillator have been considered by Stone and Holmes[22] in the case of white Gaussian noise forcing with spectral intensity ρ. They found from calculations based on the Fokker-Planck equation that the time τ between consecutive returns to a neighborhood of the saddle point is

$$\tau = c - (1/\lambda_u)\ln(\epsilon\rho),$$

where c is a constant and λ_u is the eigenvalue associated with the unstable manifold of the saddle point. τ can be shown to play a prominent role in determining the spectrum of the oscillator dynamics. Using a calculation of the phase space flux with the noise representation in Eq. (5), this expression for τ can be shown to apply not only to white noise, but more generally to the case of colored noise.[21] The phase space flux approach has the added advantage of being simpler to apply.

SYSTEMS WITH MULTIPLICATIVE EXCITATION

We turn now to a more general form for w, the multiplicative excitation model:

$$w(x, \dot{x}, t) = \gamma(x, \dot{x})g(t) + \rho(x, \dot{x})G(t). \tag{16}$$

As in the additive excitation model, the function g represents deterministic forcing while $G(t) = G(t, \omega)$, $\omega \in \Omega$ is a stochastic process representing a random forcing contribution.

The Melnikov function is calculated as in Eq. (9) to be

$$M(t_1,t_2) = \mathcal{M}[g,G] = \int_{-\infty}^{\infty} \dot{x}_s(t)[\gamma(x_s(t),\dot{x}_s(t))g(t+t_1)+\rho(x_s(t),\dot{x}_s(t))G(t+t_2)]dt.$$

We define orbit filters $\mathcal{F}_1$ and $\mathcal{F}_2$ with impulse responses

$$h_1(t) = \dot{x}_s(-t)\gamma(x_s(-t),\dot{x}_s(-t)), \quad h_2(t) = \dot{x}_s(-t)\rho(x_s(-t),\dot{x}_s(-t))$$

and corresponding transfer functions $H_1(\nu)$ and $H_2(\nu)$. Then

$$M(t_1,t_2) = \mathcal{F}_1[g](t_1) + \mathcal{F}_2[G](t_2). \tag{17}$$

Generalizing the additive excitation model Eq. (3) by allowing the coefficients γ and ρ to depend on the state $(x,\dot{x})$ of the system has, according to (17), two significant consequences. First, the orbit filter $\mathcal{F}$ in the additive model is replaced in the multiplicative model by two different orbit filters $\mathcal{F}_1$ and $\mathcal{F}_2$ and, second, the filters $\mathcal{F}_1$ and $\mathcal{F}_2$ are linear, time-invariant and noncausal with impulse responses given solely in terms of the orbit $\vec{x}_s$ of the unperturbed system and the functions γ and ρ.

Substituting Eq. (17) into Eq. (8) gives

$$\Phi = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [\rho\mathcal{F}_1[g](\theta_1 - s) + \gamma\mathcal{F}_2[G](\theta_2 - s)]^+ ds. \tag{18}$$

Just as in the case of the additive excitation model, existence of the limit in Eq. (18) hinges on the joint ergodicity of the function $\mathcal{F}_1[g] = g * h_1$ and the process $\mathcal{F}_2[G] = G * h_2$.

THEOREM 3. Consider the system governed by Eq. (2) with perturbation function w as in Eq. (16) such that g is AMS and $\mathcal{F}_2[G]$ is ergodic. Let $\mu_{\mathcal{F}_1[g]}$ and $\mu_{\mathcal{F}_2[G]}$ be the SM distributions of $\mathcal{F}_1[g]$ and $\mathcal{F}_2[G]$, respectively. Then the limit in Eq. (18) exists, the flux factor Φ is nonrandom and

$$\Phi = E[(\gamma A + \rho B)^+]$$

where A is a random variable with distribution $\mu_{\mathcal{F}_1[g]}$, B is a random variable with distribution $\mu_{\mathcal{F}_2[G]}$ and A and B are independent.

Theorem 3 is an extension of Theorem 1 to the general case of multiplicative excitation. Theorem 3 can in turn be extended to systems with more general planar vector fields than that of the system governed by Eq. (2). Only the orbit filters $\mathcal{F}_1$ and $\mathcal{F}_2$ change in these more general cases; the form of the flux factor Φ given in Theorem 3 remains the same.

DUFFING OSCILLATOR WITH SHOT NOISE-LIKE DISSIPATION

As an example of a system with multiplicative shot noise, we consider the Duffing oscillator with weak forcing and non-autonomous damping:

$$\ddot{x} = x - x^3 + \varepsilon[\gamma g(t) - \kappa(K_N(t) + \eta)\dot{x}]. \tag{19}$$

Here $\gamma \geq 0$, $\kappa \geq 0$ and $\eta \geq 0$ are constants, g is deterministic and bounded $|g(t)| \leq 1$, and K_N is a form of shot noise. The perturbation in Eq. (18) is a particular case of the multiplicative excitation model (16) with $\gamma(x, \dot{x}) = \gamma$, $\rho(x, \dot{x}) = -\kappa\dot{x}$, and $G(t) = K_N(t) + \eta$. $\kappa(K_N(t) + \eta)$ in Eq. (18) serves as a time-varying damping factor and plays the same role as the constant κ in Eq. (4). The two terms $\kappa\eta$ and κK_N represent, respectively, viscous and shot noise-like damping forces. We assume for this example that $\eta = 0$. We also assume the shot response r (see below) of K_N to be nonnegative in this example so that the factor κK_N is nonnegative.

The usual model of constant-rate shot noise is a stochastic process of the form[11,23]

$$K(t) = \sum_{k \in \mathcal{Z}} r(t - T_k), \tag{20}$$

where $\mathcal{Z}$ is the set of integers, $\{T_k, k \in \mathcal{Z}\}$ are the epochs (shots) of a Poisson process with rate $\lambda > 0$ and r is bounded and square-integrable,

$$\int_{-\infty}^{\infty} r^2(t)dt < \infty.$$

r is call the shot response of the process K.

The shot noise model K in Eq. (20) is neither bounded nor EUC and cannot be used in conjunction with Melnikov theory in calculating the phase space flux in chaotic systems. A modification of the model which approximates K and yet has the requisite path properties has been developed[8]:

$$K_N(t) = \sum_{j \in \mathcal{Z}} \sum_{k=1}^{2^N} r(t - T_{jkN} - A_j - T) \tag{21}$$

where N is a positive integer, $A_j = 2^N(j - 1/2)/\lambda$ and $\{T, T_{jkN}, j \in \mathcal{Z}, k = 1, \ldots, 2^N\}$ are independent random variables such that for each N and j, $\{T_{jkN}, k = 1, 2, \ldots, 2^N\}$ are identically uniformly distributed in the interval $(A_j, A_{j+1}]$ and T is uniformly distributed between 0 and $2^N/\lambda$. As in the usual shot noise model (20), λ is here again the rate of the process; it is the mean number of epochs (shots) T_{jkN} per unit time. We assume just as for K, that r in Eq. (21) is bounded and square-integrable. We further assume that r is UC and that the radial majorant

$$r^*(t) = \sup_{|\tau| \geq |t|} |r(\tau)|$$

of the shot response is integrable; i.e.,

$$\int_{-\infty}^{\infty} r^*(t)dt < \infty.$$

According to this specification of K_N, realizations of the process are obtained by partitioning the real line into the intervals $(A_j, A_{j+1}]$ of length $2^N/\lambda$ with common random phase T and then placing 2^N epochs independently and at random in each interval. The random phase T eliminates the (ensemble) cyclic nonstationarity produced by the partitioning by $(A_j, A_{j+1}]$.

It can be shown that K_N and $\mathcal{F}[K_N]$ are stationary processes, that K_N converges in distribution[4] to the shot noise K with the same shot response r and rate λ, that $\mathcal{F}[K_N]$ is also a shot noise of the form Eq. (21) with shot response $r * h$, and $\mathcal{F}[K_N]$ converges in distribution to the shot noise K with shot response $r * h$ and rate λ, that the variances of K_N and $\mathcal{F}[K_N]$ converge, respectively, to those of K and $\mathcal{F}[K]$, that the spectrum of K_N converges weakly[4] to the spectrum of the shot noise K with the same shot response r and rate λ, and that the spectrum of $\mathcal{F}[K_N]$ converges weakly to the spectrum of the shot noise K with shot response $r * h$ and rate λ. It can also be shown that K_N is uniformly bounded for all N, that K_N is EUC for all N, and that K_N and $\mathcal{F}[K_N]$ are each ergodic for all N. Hence, for large N the shot noise K_N closely approximates the standard shot noise model K in all important respects. Also, K_N, unlike K, can be used in Melnikov's method-type calculations of the flux factor. Additional details can be found in Frey and Simiu.[8]

According to Theorem 3, the Melnikov function for the Duffing oscillator Eq. (19) is

$$M(t_1, t_2) = \mathcal{F}_1[g](t_1) + \mathcal{F}_2[G](t_2)$$

where

$$h_1(t) = \gamma \dot{x}_s(-t) = \gamma\sqrt{2}\operatorname{sech} t \tanh t$$

and

$$h_2(t) = -\kappa \dot{x}_s^2(-t) = -2\kappa \operatorname{sech}^2 t \tanh^2 t.$$

The corresponding moduli of the filters $\mathcal{F}_1$ and $\mathcal{F}_2$ are

$$S_1(\nu) = \sqrt{2}\pi\gamma\nu \operatorname{sech}\frac{\pi\nu}{2}$$

and

$$S_2(\nu) = 4\kappa \int_0^{\infty} \operatorname{sech}^2 t \tanh^2 t \cos \nu t dt.$$

We have $S_1(0) = 0$ so the d.c. component (if any) of g is completely removed by $\mathcal{F}_1$ and has no effect on the Melnikov function. K_N does have a d.c. component; K_N is ergodic so its d.c. component is $E[K_N] = \lambda R(0)$ where

$$R(0) = \int^{\infty} r(t)dt > 0.$$

$S_2(0) = 4\kappa/3 > 0$ so the d.c. component of K_N passed by $\mathcal{F}_2$ is

$$E[\mathcal{F}_2[K_N]] = E[K_N]S_2(0) = \frac{4\kappa\lambda R(0)}{3}.$$

The presence of a d.c. component plays a pivotal role in shifting the parametric threshold for chaos. See Frey and Simiu[7] for further discussion.

Assume the deterministic forcing function g is AMS. K_N is uniformly bounded and EUC and $\mathcal{F}_2[K_N]$ is ergodic. Thus $\mathcal{F}_1[g]$ and $\mathcal{F}_2[K_N]$ are jointly ergodic. By Theorem 3, the flux factor Φ exists and

$$\Phi = E[(A - B_N)^+]$$

where the distribution of A is $\mu_{\mathcal{F}_1[g]}$, the distribution of B_N is $\mu_{\mathcal{F}_2[K_N]}$ and A and B_N are independent.

We noted earlier that the distribution of $\mathcal{F}_2[K_N]$ is, for large N, approximately that of the shot noise $\mathcal{F}_2[K]$. This is the basis for the following theorem.

THEOREM 4. The flux factor Φ for the Duffing oscillator (19) with weak forcing and shot noise damping coefficent κK_N is approximately

$$\Phi \doteq E[(A - B)^+]$$

where A is $\mu_{\mathcal{F}_1[g]}$-distributed, B is $\mu_{\mathcal{F}_2[K]}$-distributed, A and B are independent and K is the shot noise (20). This approximation improves as N increases.

Φ can be calculated numerically as follows for given system parameters ν, γ and κ and shot parameters λ and r. Make the following definitions:

$$\Phi' = \frac{\Phi}{\gamma S_1(\nu)}, \quad A' = \frac{A}{\gamma S_1(\nu)}, \quad B' = \frac{B_N}{\gamma S_1(\nu)},$$
$$\lambda' = \frac{16\lambda R^2(0)}{9J}, \quad \kappa' = \frac{3\kappa J}{4\gamma S_1(\nu) R(0)},$$

where

$$J = \int_{-\infty}^{\infty} (r * h)^2(t)dt.$$

Then

$$\Phi' = E[(A' - B')^+]. \tag{22}$$

The random variable B' is approximately gamma-distributed[16] with density

$$\frac{t^{\alpha-1}e^{-t/\beta}}{\beta^\alpha \Gamma(\alpha)}, \quad t > 0$$

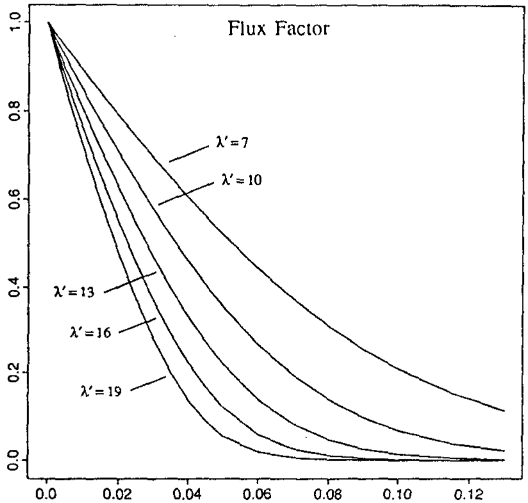

FIGURE 3 The flux factor Φ' as a function of the damping constant κ' for various shot rates λ'.

where the parameters α and β are determined by the condition that $E[B']$ and $Var[B']$ equal the mean and the variance, respectively, of the gamma distribution. $\Phi' = \Phi'(\kappa', \lambda')$ is plotted in Figure 3. Additional details are available from Frey and Simui.[8]

SUMMARY

Noise can cause multistable dynamical systems to exhibit chaotic motion with sensitive dependence upon initial conditions. The theory applicable to noise-induced chaotic dynamics reviewed in this paper rests primarily on the concept of the Melnikov transform and on techniques for approximating noise with any given spectrum and marginal distribution by uniformly bounded, ensemble uniformly continuous processes. The results described here apply to weakly perturbed, one-degree-of-freedom dynamical systems featuring homoclinic or heteroclinic orbits. Results were first given for additive perturbation and then generalized to multiplicative perturbation. Extensions of this work to higher-degree-of-freedom systems and to spatially extended systems are in progress.

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